Q.1: According to Newton's Law of Gravitation we know the force F exerted by a body of mass m_1 on a body of mass m_2 is given by the formula

$$F = \frac{G m_1 m_2}{r^2}$$

where G is the gravitational constant, m_1 & m_2 are constant & r is the distance between the bodies.

(a) Find F' with respect to the distance r.

(b) What does the -ve sign in the expression of F' mean?

(c) Assume planet Mars attracts another body at a rate of 5N/km when it is 10000 km away. How fast is F changing when the body is 7000 km away.

Sol^n:- Here $F = \frac{G m_1 m_2}{r^2}$

(a) Then derivative of F wrt r is,

$$\frac{dF}{dr} = \frac{d}{dr}\left[\frac{G m_1 m_2}{r^2}\right] = G m_1 m_2 \frac{d}{dr}\left(\frac{1}{r^2}\right) = G m_1 m_2 \left(-\frac{2}{r^3}\right) = \frac{-2 G m_1 m_2}{r^3}$$

(b) The -ve sign in $\frac{dF}{dr}$ implies the exerted force F decreases as the distance between the bodies increases.

(c) Given mars attracts a body at a rate of 5N/km.

ie, $\frac{dF}{dr} = 5$ at $r = 10000$

So, $-\frac{2Gm_1m_2}{(10,000)^3} = 5$

$\Rightarrow -2Gm_1m_2 = 5 \times 10^{12}$

Now we need to find $\frac{dF}{dr}$, when $r = 7,000$.

Therefore,
$$\frac{dF}{dr} = \frac{-2Gm_1m_2}{(7000)^3}$$

$$= \frac{5 \times 10^{12}}{(7 \times 10^3)^3}$$

$$= \frac{5 \times 10^{12}}{7^3 \times 10^9}$$

$$= \frac{5 \times 10^3}{7^3}$$

$$= \frac{5000}{7^3} \approx 14.57 \text{ N/km.}$$

Q.2: Find the differential of $y = e^{4x} + \ln\left(\frac{1}{x^7}\right)$

Sol^n:- We know differential of $y = f(x)$ is given by

$$dy = f'(x) \cdot dx$$

So we need to find $f'(x)$; where $f(x) = e^{4x} + \ln\left(\frac{1}{x^7}\right)$

Now, since we have log/ln in the expression, we first try to simplify.

$$f(x) = e^{4x} + \ln\left(\frac{1}{x^7}\right)$$

$$= e^{4x} + \left[\ln(1) - \ln(x^7)\right]$$

$$= e^{4x} + \left[0 - 7\ln x\right]$$

$$= e^{4x} - 7\ln x$$

Hence, $f'(x) = \frac{d}{dx}\left[e^{4x} - 7\ln x\right]$

$$= \frac{d}{dx}(e^{4x}) - \frac{d}{dx}(7\ln x)$$

$$= e^{4x} \cdot 4 - 7 \cdot \frac{1}{x}$$

$$= 4e^{4x} - \frac{7}{x}$$

Therefore, $dy = \left(4e^{4x} - \frac{7}{x}\right) dx$

Q.3. Find the value of $\sin(31^\circ)$, using linear approximation.

Sol^n:- Note it is not given, at which place we need to find the linear approximation.

So by observation we pick a suitable 'a'.

Since $\sin(30^\circ) = \frac{1}{2}$, we can use this & pick our

$a = 30^\circ$ & $f(x) = \sin x$.

Now, $f'(x) = \cos x \Rightarrow f'(30^\circ) = \cos(30^\circ) = \frac{\sqrt{3}}{2}$

So, the Linear approximation of $f(x)$ at $a = 30^\circ$ is given by

$$L(x) = f(30^\circ) + f'(30^\circ).(x - 30^\circ)$$
$$= \frac{1}{2} + \frac{\sqrt{3}}{2}(x - 30^\circ)$$

Now we need to find $L(31^\circ)$, so plugging in $x = 31^\circ$ in $L(x)$ we get,

$$L(31^\circ) = \frac{1}{2} + \frac{\sqrt{3}}{2}(31^\circ - 30^\circ) = \frac{1}{2} + \frac{\sqrt{3}}{2}(1^\circ)$$
$$= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{\pi}{180}\right) = \frac{1}{2} + \frac{\sqrt{3}\pi}{360}$$

Q.4 The position of the object at time t is given by $s(t) = -4t^3 + 5e^{2t}$. What is the velocity of the object at $t=4$?

Sol^n:- $s(t) = -4t^3 + 5e^{2t}$

Then $v(t) = s'(t) = \frac{d}{dt}\left[-4t^3 + 5e^{2t}\right]$

$= -4(3t^2) + 5(e^{2t} \cdot 2)$

$= -12t^2 + 10e^{2t}$

So, $v(4) = -12(4)^2 + 10e^{2(4)}$

$= -192 + 10e^8$

$= 10e^8 - 192$

Q.5 Let $g(x) = x^3 f(x)$, then find $g''(x)$.

Sol^n:- $g(x) = x^3 f(x)$

$\Rightarrow g'(x) = \frac{d}{dx}[g(x)]$

$= \frac{d}{dx}[x^3 f(x)]$

$= \frac{d}{dx}(x^3) f(x) + x^3 \cdot \frac{d}{dx}[f(x)]$

$= 3x^2 f(x) + x^3 f'(x)$

& $$g''(x) = \frac{d}{dx}[g'(x)]$$

$$= \frac{d}{dx}[3x^2 f(x) + x^3 f'(x)]$$

$$= \frac{d}{dx}[3x^2 f(x)] + \frac{d}{dx}[x^3 f'(x)]$$

$$= 3\frac{d}{dx}[x^2 f(x)] + \frac{d}{dx}[x^3 f'(x)]$$

$$= 3\left\{\frac{d}{dx}[x^2] f(x) + x^2 \frac{d}{dx}[f(x)]\right\} + \left\{\frac{d}{dx}[x^3] f'(x) + x^3 \frac{d}{dx}[f'(x)]\right\}$$

$$= 3\{2x f(x) + x^2 f'(x)\} + 3x^2 f'(x) + x^3 f''(x)$$

$$= 6x f(x) + 3x^2 f'(x) + 3x^2 f'(x) + x^3 f''(x)$$

$$= 6x f(x) + 6x^2 f'(x) + x^3 f''(x)$$

Q.6. Find the asymptotes for $f(x) = \frac{\sqrt{6x^2+7}}{x+7}$

Sol^n: **Note:** denominator is 0 when $x = -7$.

So if we choose $\lim_{x \to -7} f(x) = \lim_{x \to -7} \frac{\sqrt{6x^2+7}}{x+7} = \infty$

Hence, $x = -7$ is the Vertical Asymptote.

Also, $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\sqrt{6x^2+7}}{x+7}$

$$= \lim_{x \to \infty} \frac{\sqrt{x^2\left(6+\frac{7}{x^2}\right)}}{x+7}$$

$$= \lim_{x \to \infty} \frac{x\sqrt{6+\frac{7}{x^2}}}{x\left(1+\frac{7}{x}\right)}$$

$$= \lim_{x \to \infty} \frac{\sqrt{6 + \frac{7}{x^2}}}{1 + \frac{7}{x}}$$

$$= \frac{\sqrt{6+0}}{1+0} = \sqrt{6}.$$

Hence, $y = \sqrt{6}$ is a Horizontal Asymptote.

* If $f'(c)$ exists, then it measures the instantaneous rate of change at $x = c$.

* If f is continuous on a closed & bounded interval, then f attain both its absolute maxima & absolute minima.